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Scalar Gravity and Higgs Potential

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A general Lorentz-invariant scalar gravitational interaction theory with self-interaction is presented. It is shown that this theory leads to the recently proposed Higgs-field gravity and thereby provides a new approach to the Higgs potential.

1. INTRODUCTION

Scalar theories of elementary particles and their interactions are of interest due to their importance as Higgs fields in the theory of spontaneous symmetry breaking. In addition, scalar theories of gravitation have a long history; the classical example is the Newtonian theory of gravity, but also more modern theories, for example, the Dicke-Brans-Jordan theory (Brans and Dicke, 1961) and the examples in Misner *et al.* (1973, p. 178, Exercise 7.1, and p. 1070) deal with scalar interactions.

Recently we have pointed out (Dehnen and Frommert, 1990; Dehnen *et al.*, 1990) that the scalar interaction mediated by the Higgs field in theories with spontaneous symmetry breaking is of gravitational type, i.e., it is coupled to the masses of the elementary particles and not to any other charges: Mass, not some current, is the source of the scalar Higgs field, and the Higgs field acts back by its gradient on the mass in the momentum law. Moreover, the spontaneous symmetry breaking generates exactly the mass of the elementary particles, which then serves as active and passive gravitational mass. Thereby, Einstein's "principle of relativity of inertia" [Mach's principle; see Einstein (1917)] is fulfilled: Mass is generated by the same mechanism as the gravitational interaction. In this sense, the inertial mass as a measure for the resistance of a particle against the relative acceleration with respect to other particles has its origin in the gravitational interaction with all other particles in the universe.

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Here I go the opposite way and construct a very general scalar gravitational theory between elementary particles on the level of special relativity. It is imposed to contain self-interaction, and to obey a Yukawa-type field equation, i.e., I consider a massive scalar field. This is even necessary, because the only source of Lorentz-invariant scalar gravity is the trace of the energy-momentum tensor, which vanishes in the massless case. I show that the scalar field equation is exactly that of the Higgs field with the correct Higgs potential.

2. STRUCTURE OF THE LAGRANGE DENSITY

The most general Lagrange density of a pure scalar field φ containing the derivatives of φ at most quadratically is given by (sign +, −, −, −)

$$L_0 = L_0(\partial_\lambda \varphi, \varphi) = \tfrac{1}{2}(\partial_\lambda \varphi)\partial^\lambda \varphi - V(\varphi) \tag{2.1}$$

where $V(\varphi)$ is some arbitrary functional of the field φ, usually called the potential term of L_0.

To construct a theory of scalar *interaction*, a "matter" term L_M and an "interaction" part L_{int} must be added to (2.1) in order to obtain the complete Lagrange density:

$$L = L_0 + L_M + L_{\text{int}} \tag{2.2}$$

where L_M is the Lorentz-invariant Lagrange density of the pure matter fields ψ^A (A represents some set of inner, spinor, or tensor indices which are not specified here) and L_{int} is the interaction part, depending on φ and ψ^A only and not on their derivatives:

$$L_M = L_M(\partial_\lambda \psi^A, \psi^A) \tag{2.3}$$

$$L_{\text{int}} = L_{\text{int}}(\varphi, \psi^A) \tag{2.4}$$

The field equations for φ and ψ^A obtained from (2.2) by the variational principle are

$$\partial_\lambda \partial^\lambda \varphi + \frac{\partial V}{\partial \varphi} = \frac{\partial L_{\text{int}}}{\partial \varphi} := -\eta(\varphi, \psi^A) \tag{2.5}$$

$$\partial_\lambda p_A^\lambda - \left(\frac{\partial L_M}{\partial \psi^A} + \frac{\partial L_{\text{int}}}{\partial \psi^A}\right) = 0 \tag{2.6}$$

with the *source* $\eta(\varphi, \psi^A)$ of the scalar field φ and the canonical momentum of the matter field

$$p_A^\lambda := \frac{\partial L_M}{\partial(\partial_\lambda \psi^A)} \tag{2.7}$$

The canonical energy-momentum tensor is given by

$$T^{\mu}_{\lambda} = T^{\mu}_{\lambda}(\varphi) + T^{\mu}_{\lambda}(\psi) \tag{2.8}$$

where

$$T^{\mu}_{\lambda}(\varphi) = (\partial_{\lambda}\varphi)\partial^{\mu}\varphi - \delta^{\mu}_{\lambda}[\tfrac{1}{2}(\partial_{v}\varphi)\partial^{v}\varphi - V(\varphi)] \tag{2.8a}$$

and

$$T^{\mu}_{\lambda}(\psi) = p^{\mu}_{A}\partial_{\lambda}\psi^{A} - \delta^{\mu}_{\lambda}(L_M + L_{\text{int}}) \tag{2.8b}$$

are the parts of T^{μ}_{λ} resulting from the pure scalar field φ and the matter fields ψ^{A}, respectively. It obeys, with respect to the field equations (2.5) and (2.6), the equation of continuity:

$$\partial_{\mu}T^{\mu}_{\lambda} = 0 \tag{2.9}$$

and has the trace

$$T = T^{\lambda}_{\lambda} = T(\varphi) + T(\psi) = [-(\partial_{\lambda}\varphi)\partial^{\lambda}\varphi + 4V(\varphi)] + [p^{\lambda}_{A}\partial_{\lambda}\psi^{A} - 4(L_M + L_{\text{int}})] \tag{2.10}$$

which represents the rest mass-energy densities of the scalar field and the matter field, respectively.

Splitting T^{μ}_{λ} according to (2.8), the equation of continuity (2.9) yields

$$0 = \partial_{\mu}T^{\mu}_{\lambda}(\varphi) + \partial_{\mu}T^{\mu}_{\lambda}(\psi) = -(\partial_{\lambda}\varphi)\eta + \partial_{\mu}T^{\mu}_{\lambda}(\psi) \tag{2.11}$$

where equation (2.8a) and the field equation of the scalar field (2.5) are inserted. Obviously, equation (2.11) can be rewritten as

$$\partial_{\mu}T^{\mu}_{\lambda}(\psi) = (\partial_{\lambda}\varphi) \cdot \eta \tag{2.11a}$$

By integration over a spacelike hypersurface one obtains, neglecting boundary terms on the left-hand side of (2.11a),

$$\frac{d}{dt}P_{\lambda} := \frac{\partial}{\partial t}\int d^3x\, T^{0}_{\lambda}(\psi) = \int d^3x\, (\partial_{\lambda}\varphi)\eta := K_{\lambda} \tag{2.12}$$

which is the *momentum law* for the matter field: The 4-momentum P_{λ} of the matter fields on the left-hand side of (2.12) is changed with time by the 4-force K_{λ} caused by the 4-gradient of the scalar field φ acting on the matter field described by η, which is simultaneously the source of the scalar field φ according to (2.5). As it must be, particles that do not participate in the interaction are not influenced by the scalar force, due to $\eta = 0$ in this case.

Evidently equations (2.5) and (2.12) describe a self-consistent *gravitational* interaction only if η is proportional to the trace $T(\psi)$ of the energy-momentum tensor of the matter field according to (2.10).

3. DETERMINATION OF THE POTENTIAL

Now, in a *physical* theory, the rest mass-energy must have a lower bound in order to avoid infinite negative energies, and, therefore, according to (2.10), the potential term $V(\varphi)$ should have a minimum, say, at $\varphi = v$. Expanding the potential in the neighborhood of this minimum, one obtains

$$V(\varphi) = V_0 + \frac{1}{2}\left(\frac{M}{\hbar}\right)^2 (\varphi - v)^2 + \mathcal{O}[(\varphi - v)^3] \tag{3.1}$$

with $V_0 = V(v)$ and $M = \text{const}$. At this stage we assume $v \neq 0$.[2] Furthermore it is convenient to introduce the *excited scalar field* χ according to

$$\varphi = v(1+\chi) \tag{3.2}$$

With the *new source*

$$\hat{\eta} = -\frac{\partial L_{\text{int}}}{\partial \chi} = v\eta \tag{3.3}$$

and the *new potential*

$$\hat{V} = \hat{V}(\chi) = \frac{V(\chi)}{v^2} = \frac{V_0}{v^2} + \frac{1}{2}\left(\frac{M}{\hbar}\right)^2 \chi^2 + \mathcal{O}(\chi^3) \tag{3.4}$$

one obtains from (2.5) the field equations for the excited scalar field χ,

$$\partial_\lambda \partial^\lambda \chi + \frac{\partial \hat{V}}{\partial \chi} = -\frac{1}{v^2}\hat{\eta} \tag{3.5}$$

with

$$\frac{\partial \hat{V}}{\partial \chi} = \left(\frac{M}{\hbar}\right)^2 \chi + \mathcal{O}(\chi^2) \tag{3.6}$$

according to which M is the mass of the excited scalar field χ. Analogously, the new equation of continuity [see (2.11a)] reads

$$\partial_\mu T^\mu_\lambda(\psi) = (\partial_\lambda \chi) \cdot \hat{\eta} \tag{3.7}$$

and the momentum law (2.12) takes the new form

$$\frac{d}{dt} P_\lambda = \frac{\partial}{\partial t} \int d^3x\, T^0_\lambda(\psi) = \int d^3x\, (\partial_\lambda \chi)\hat{\eta} = K_\lambda \tag{3.8}$$

Comparison with Newtonian gravity shows that $1/v^2$ plays the role of the gravitational constant.

[2]It should be noted that in the case of $v = 0$ no meaningful scalar gravity can be constructed.

For the establishment of the *gravitational* character of the scalar interaction in detail, it remains (see the last remark of Section 2) to postulate that the source $\hat{\eta}$ in (3.5), see also (3.8), is proportional to the trace $T(\psi)$ of the energy-momentum tensor of the matter fields. This means

$$\hat{\eta} = F(\chi)T(\psi) \tag{3.9}$$

with some functional $F(\chi)$. To realize the gravitational *self-interaction*, add $F(\chi)T(\chi)/v^2$ on both sides of (3.5), where

$$T(\chi) = T(\varphi) \tag{3.10}$$

In this way we get from (3.5) the field equation for the scalar field in the form

$$\partial_\lambda \partial^\lambda \chi + \frac{\partial \hat{V}}{\partial \chi} - \frac{F(\chi)}{v^2} T(\chi) = -\frac{F(\chi)}{v^2}[T(\psi) + T(\chi)] \tag{3.11}$$

Simultaneously, the momentum law (3.8) reads

$$\frac{d}{dt} P_\lambda = \int d^3x \, (\partial_\lambda \chi) F(\chi) T(\psi) \tag{3.12}$$

Dividing (3.11) by $F(\chi)$, we obtain

$$\frac{1}{F(\chi)} \partial_\lambda \partial^\lambda \chi + \frac{1}{F(\chi)} \frac{\partial \hat{V}}{\partial \chi} - \frac{1}{v^2} T(\chi) = -\frac{1}{v^2}[T(\psi) + T(\chi)] \tag{3.13}$$

In case of a self-interacting scalar gravity, a functional $u(\chi)$ should exist in such a way that equation (3.13) takes the form ($a = \text{const}$)

$$\partial_\lambda \partial^\lambda u + a^2 u = -\frac{1}{v^2}[T(\psi) + T(u)] \tag{3.14}$$

This is a Yukawa equation with the mass term $a^2 u$ and self-interaction described by the term $T(u)$; for $T(u)$ one obtains from (2.10) and (3.10) using (3.2)

$$T(u) = T(\chi) = -v^2[(\partial_\lambda \chi)\partial^\lambda \chi - 4\hat{V}(\chi)] \tag{3.15}$$

Inserting the identity

$$\partial_\lambda \partial^\lambda u(\chi) = \frac{\partial u}{\partial \chi} \partial_\lambda \partial^\lambda \chi + \left(\frac{\partial^2 u}{\partial \chi^2}\right)(\partial_\lambda \chi)\partial^\lambda \chi \tag{3.16}$$

into (3.14) and subtracting (3.13) after insertion of (3.15), we find

$$\left(\frac{\partial u}{\partial \chi} - \frac{1}{F}\right)\partial_\lambda \partial^\lambda \chi + \left(\frac{\partial^2 u}{\partial \chi^2} - 1\right)(\partial_\lambda \chi)\partial^\lambda \chi - \left(\frac{1}{F}\frac{\partial \hat{V}}{\partial \chi} - 4\hat{V} - a^2 u\right) = 0 \tag{3.17}$$

This equation must hold for arbitrary $\partial_\lambda \chi, \partial_\lambda \partial^\lambda \chi$; this requires that each of the three terms in (3.17) vanishes independently, resulting in

$$\frac{\partial^2 u}{\partial \chi^2} = 1 \tag{3.18a}$$

$$\frac{1}{F}\frac{\partial \hat{V}}{\partial \chi} - 4\hat{V}(\chi) - a^2 u(\chi) = 0 \tag{3.18b}$$

$$F(\chi) = \frac{1}{\partial u/\partial \chi} \tag{3.18c}$$

Integration of (3.18a) determines u up to two constants of integration A and B:

$$u = \tfrac{1}{2}(\chi^2 + 2A\chi + B) = \tfrac{1}{2}[(\chi + A)^2 + (B - A^2)] \tag{3.19}$$

From this one has

$$\frac{\partial u}{\partial \chi} = \chi + A \tag{3.19a}$$

and (3.18c) gives the functional $F(\chi)$:

$$F(\chi) = \frac{1}{\chi + A} \tag{3.20}$$

by which (3.18b) yields after integration

$$\hat{V}(\chi) = \hat{V}_0 + \frac{C}{2} a^2 \left[(\chi + A)^2 - \frac{1}{4C}\right]^2 \tag{3.21}$$

with

$$\hat{V}_0 = \tfrac{1}{8} a^2 \left(A^2 - B - \frac{1}{4C}\right) \tag{3.21a}$$

where C is the new integration constant. Herewith, the demanded structure of (3.14) is achieved.

Now, in order to specify the potential $\hat{V}$ or V explicitly, the constants A, B, C, and a must be determined. Remembering that $\hat{V}$ has a minimum at $\chi = 0$ results in the relation

$$C = \frac{1}{4A^2} \tag{3.22}$$

Herewith $\hat{V}_0$ is simplified to

$$\hat{V}_0 = -\frac{B}{8} a^2 \tag{3.22a}$$

and $\hat{V}(\chi)$ takes the form

$$\hat{V}(\chi) = \hat{V}_0 + \frac{1}{2}\left(\frac{Aa}{2}\right)^2\left[\left(1+\frac{\chi}{A}\right)^2 - 1\right]^2 \tag{3.23}$$

Comparison of (3.23) with (3.4) gives

$$a = \frac{M}{\hbar} \tag{3.24}$$

so that M represents also the mass of the field u [cf. (3.14)].

The integration constant A lacks a deeper physical meaning because it can be eliminated by the substitution of the scalar field φ by a *new* one ϕ differing from φ by a constant only:

$$\varphi \to \phi = \varphi + (A-1)v \tag{3.25}$$

ϕ obeys the same field equation (2.5) as φ because of $\partial V/\partial\phi = \partial V/\partial\varphi$:

$$\partial_\lambda \partial^\lambda \phi + \frac{\partial V}{\partial \phi} = -\eta \tag{3.26}$$

The use of ϕ instead of φ leads to a minimum of $V(\phi)$ at

$$\phi = v' = \phi(\varphi = v) = Av \tag{3.27}$$

The new excited scalar field is

$$\chi' = \frac{\phi - v'}{v'} = \frac{\chi}{A} \tag{3.28}$$

which obeys a field equation of the form of (3.5):

$$\partial_\lambda \partial^\lambda \chi' + \frac{\partial \hat{V}'}{\partial \chi'} = -\frac{1}{v'^2}\hat{\eta}' \tag{3.29}$$

where

$$\hat{V}'(\chi') = \frac{V}{v'^2} = \frac{\hat{V}}{A^2} \tag{3.29a}$$

and

$$\hat{\eta}' = v'\eta = A\hat{\eta} = -\frac{\partial L_{\text{int}}}{\partial \chi'} \tag{3.29b}$$

With the use of the relations (3.24), (3.28), and (3.29a), the following expression for $\hat{V}'$ follows from equation (3.23):

$$\hat{V}'(\chi') = \hat{V}_0' + \frac{1}{8}\left(\frac{M}{\hbar}\right)^2 [(1+\chi')^2 - 1]^2 \tag{3.30}$$

where

$$\hat{V}'_0 = \frac{\hat{V}_0}{A^2} = -\frac{1}{8}\frac{B}{A^2}\left(\frac{M}{\hbar}\right)^2 \tag{3.30a}$$

Obviously only the quotient B/A^2 appears in the potential (3.30).

Now one is able to write down the potential $V(\phi)$ in the field equation (3.26) explicitly; insertion of (3.30) into (3.29a) and resolution with respect to V give, with the use of (3.28),

$$V(\phi) = \frac{v'^2}{8}\left(1 - \frac{B}{A^2}\right)\left(\frac{M}{\hbar}\right)^2 - \frac{1}{4}\left(\frac{M}{\hbar}\right)^2 \phi^2 + \frac{1}{8}\left(\frac{M}{\hbar}\right)^2 \frac{1}{v'^2}\phi^4 \tag{3.31}$$

The minimum lies at $\phi = v'$ and this minimum value is

$$V(v') = -\frac{B}{A^2}\left(\frac{M}{\hbar}\right)^2 \frac{v'^2}{8}$$

The quantity B/A^2 has no deeper meaning because it is contained only in the additive constant of the potential (3.31) or in its minimum value. For simplicity, I choose without restriction of generality the additive constant of the potential $V(\phi)$ to be equal to zero; this means

$$\frac{B}{A^2} = 1 \tag{3.31a}$$

Evidently, equation (3.31) represents the Higgs potential, which is here derived by postulating a scalar self-interacting massive gravitational field. Moreover, the field equation (3.26) for the scalar field ϕ is exactly the Higgs field equation.

4. SCALAR GRAVITY WITH SELF-INTERACTION

Finally I give an explicit representation of the self-interacting massive scalar gravity introduced in Section 3. In view of the ground state v' for the scalar field ϕ according to (3.27) and the excited scalar field χ' given by (3.28), one can rewrite the momentum law (3.12) and the Yukawa field equation (3.14) with the use of (3.24) as follows:

$$\frac{d}{dt}P_\lambda = \int d^3x\,(\partial_\lambda \chi')F'(\chi')T(\psi) \tag{4.1}$$

and

$$\partial_\lambda \partial^\lambda u' + \left(\frac{M}{\hbar}\right)^2 u' = -\frac{1}{v'^2}[T(\psi) + T(u')] \tag{4.2}$$

$[T(u') = T(\chi')]$ with

$$u' = \frac{v^2}{v'^2} u = \frac{1}{A^2} u = \tfrac{1}{2}(1+\chi')^2 \tag{4.3a}$$

and

$$F'(\chi') = AF(\chi') = \frac{1}{1+\chi'} \tag{4.3b}$$

using (3.19), (3.20), and (3.31a). With respect to the 4-force on the right-hand side of (4.1) it is convenient to choose, instead of χ',

$$\zeta = \ln(1+\chi') \tag{4.4}$$

as new excited scalar field. It follows that $u' = \frac{1}{2}e^{2\zeta}$ and equations (4.1) and (4.2) take the form

$$\frac{d}{dt} P_\lambda = \int d^3x\, (\partial_\lambda \zeta) T(\psi) \tag{4.5}$$

$$\partial_\lambda \partial^\lambda e^{2\zeta} + \left(\frac{M}{\hbar}\right)^2 e^{2\zeta} = -\frac{2}{v'^2}[T(\psi) + T(\zeta)] \tag{4.6}$$

where $T(\zeta) = T(\chi')$. Evidently, these equations describe a massive scalar gravitational interaction with self-interaction, where, with respect to the Newtonian limit (linearization in ζ),

$$\frac{1}{v'^2} = 4\pi G\gamma \tag{4.6a}$$

has the meaning of the gravitational constant (G is the Newtonian gravitational constant) and γ is a numerical factor comparing the strength of the scalar gravity in question with the Newtonian one. Furthermore, in equation (4.6), M is the mass of the excited scalar field ζ; simultaneously there appears a cosmological constant $\frac{1}{2}(M/\hbar)^2$, which, however, drops out against the trace of the energy-momentum tensor of the ground state $T(\zeta = 0)$.

It may be of interest that a special relativistic version of the Newtonian gravity is included in the general theory for the special case

$$v'^{-2} = 4\pi G \qquad (\gamma = 1) \tag{4.7a}$$

and

$$M < \frac{\hbar}{c\cdot(10^4 \text{ light-years})} \approx 10^{-26}\left(\frac{eV}{c^2}\right) \tag{4.7b}$$

so that the range of this scalar gravity is at least 10^4 light-years, because there is experimental evidence that Newton's law of gravitation is valid at least up to this distance.

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